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Some implicative topological quasi-Boolean algebras and rough set models

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ABSTRACT

In this paper a number of implicative topological algebras have been developed which are based on the implicative quasi-Boolean algebra with operator (IqBaO) [10]. Their independence is established by several examples. Logics corresponding to these algebras are presented. Two new pairs of lower-upper approximations of a set have been introduced in order to develop the notion of duality with respect to the quasi-complementation. Set theoretic rough set models of some of the algebras are constructed using these lower-upper approximations and quasi-complementation.

1. Introduction

Quasi-Boolean algebra [8] arises naturally as the algebra of rough sets [1,6]. A quasi-Boolean algebra **(qBa)** is a structure as defined below. The algebra $\langle S, \vee, \wedge, \neg, 0, 1\rangle$ is a qBa if and only if

1. $\langle S, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice
2. $\neg\neg x = x$, for all x in S
3. $\neg(x \vee y) = \neg x \wedge \neg y$, for all x, y in S.

It is to be noted that a qBa may not be a Boolean algebra as $x \wedge \neg x \neq 0$ and hence $x \vee \neg x \neq 1$. Example 2.5 given in the sequel is not a Boolean algebra but a quasi-Boolean algebra.

In this paper, we need the definition of abstract pre-rough algebra [2] which is based on qBa. A pre-rough algebra is a structure $\langle S, \vee, \wedge, \neg, O, 0, 1\rangle$, where O is a unary operation on S with the following conditions:

1. $\langle S, \vee, \wedge, \neg, 0, 1\rangle$ is a qBa.
2. $O1 = 1$.
3. $O(x \wedge y) = Ox \wedge Oy$, for all $x, y \in S$.
4. $Ox \leq x$, for all $x \in S$ where $\leq$ is the lattice order.
5. $OOx = Ox$, for all $x \in S$.
6. $MOx = Ox$, for all $x \in S$, where $Mx = \neg O \neg x$.

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7. $\neg Ox \vee Ox = 1$, for all $x \in S$.
8. $O(x \vee y) = Ox \vee Oy$, for all $x, y \in S$.
9. $Mx \leq My$ and $Ox \leq Oy$ imply $x \leq y$, for all $x, y \in S$.

In the same paper [2], an algebra called topological quasi-Boolean algebra is defined satisfying the conditions from 1 to 6.

In [9], the definition of pre-rough algebra has further been simplified. The authors of [9] divide the original definition of topological quasi-Boolean algebra into two notions namely topological quasi-Boolean algebra **(tqBa)** satisfying conditions $1-5$ and topological quasi-Boolean algebra 5 **(tqBa5)** satisfying conditions $1-6$ stated above. It was established in [4] that in qBa (even in tqBa5) an implication operator $\rightarrow$ satisfying the property:

$$x \leq y \text{ if and only if } x \rightarrow y = 1 \qquad \textbf{(P}_{\rightarrow}\textbf{)}$$

can not be defined in terms of other operations present in qBa (even in tqBa5) (See Example 2.36). However, in pre-rough algebra such an operator can be defined by:

$$x \rightarrow y \equiv (\neg Ox \vee Oy) \wedge (\neg Mx \vee My).$$

It was observed afterwards [9] that in some algebras weaker than pre-rough algebra and stronger than qBa, the above implication operator is available. However, in [10] the implication operator has been imposed in qBa and some other stronger structures where this operator is not available in general. Following Rasiowa [8] these structures have been named implicative quasi-Boolean algebra **(IqBa)** and implicative quasi-Boolean algebra with operation **(IqBaO)**. The operations they [10] have taken are topological operators corresponding to the modal axioms T, S_4 and S_5 [5]. The corresponding algebras have been named as implicative quasi-Boolean algebra with modal axiom T**(IqBaT)**, implicative quasi-Boolean algebra with modal axiom S_4**(IqBa4)** and implicative quasi-Boolean algebra with modal axiom S_5**(IqBa5)**.

It is to be noted that in the passage from tqBa to pre-rough algebra, three intermediate properties *viz.* $\neg Ox \vee Ox = 1$, for all x **(IP1)**, $O(x \vee y) = Ox \vee Oy$, for all x, y **(IP2)** and $Mx \leq My$ and $Ox \leq Oy$ imply $x \leq y$, for all x, y **(IP3)** together play a crucial role. They are not independent in the sense that tqBa + IP1 + IP3 implies IP2 [13,9]. Three intermediate algebras using these properties are defined in [10]: by adding IP1, 1P2 and IP3 separately to tqBa5. These are named as intermediate algebra of type 1 **(IA1)**, intermediate algebra of type 2 **(IA2)** and intermediate algebra of type 3 **(IA3)** respectively.

We are now interested in adding IP1, IP2 and IP3 separately to IqBaO, IqBaT, IqBa4 and IqBa5 and investigate the consequences. For a clear understanding of the various algebraic structures obtained thus, we refer to Fig. 1 on page 3. Of these, the chain of algebras qBa, IqBa, IqBaO, IqBaT, IqBa4 and IqBa5 are included in [10]. In fact, we have actually added the modal axiom T to IqBa1 to obtain IqBa1,T which is the same as adding IP1 to IqBaT. Similar is the case for all other structures.

In addition, rough set models of some of the algebras have been presented in Section 4. A detailed discussion on rough set models is available in this section.

Section wise details of this paper are as follows.

In Section 2, a bunch of implicative topological algebras based on IqBaO is defined and their independence is established with the help of several examples. Section 3 deals with Hilbert type axiomatic systems for these algebraic structures. In Section 4, two new types of lower/upper approximations of a set have been defined which are dual approximations with respect to the quasi-complementation. Set theoretic rough set models of some of the algebras have been presented with the help of these lower-upper approximations and quasi-complementation. Section 5 contains some concluding remarks.

2. Implicative topological algebras and their independence

Quasi-Boolean algebra has been defined in the introduction. Proposition 2.1 below gives some of its properties. As stated in the introduction, the algebraic structures that will be discussed in this paper are shown in Fig. 1.

Proposition 2.1. *[8] The following properties hold in a quasi-Boolean algebra $\langle S, \vee, \wedge, \neg, 0, 1\rangle$.*

- $\neg 1 = 0$ *and* $\neg 0 = 1$.
- $\neg(x \wedge y) = \neg x \vee \neg y$, *for all* $x, y \in S$.
- $x \leq y$ *if and only if* $\neg y \leq \neg x$, *for all* $x, y \in S$, *where the lattice order* $\leq$ *is defined as* $x \leq y$ *if and only if* $x \vee y = y$ $(x \wedge y = x)$.

Definition 2.2. [10] An abstract algebra $\langle S, \vee, \wedge, \rightarrow, \neg, 0, 1\rangle$ is called an implicative quasi-Boolean algebra (IqBa) if and only if

1. $\langle S, \vee, \wedge, \neg, 0, 1\rangle$ is a qBa.
2. $x \rightarrow y = 1$ if and only if $x \leq y$, for all $x, y \in S$. $\qquad$ $\textbf{(P}_{\rightarrow}\textbf{)}$

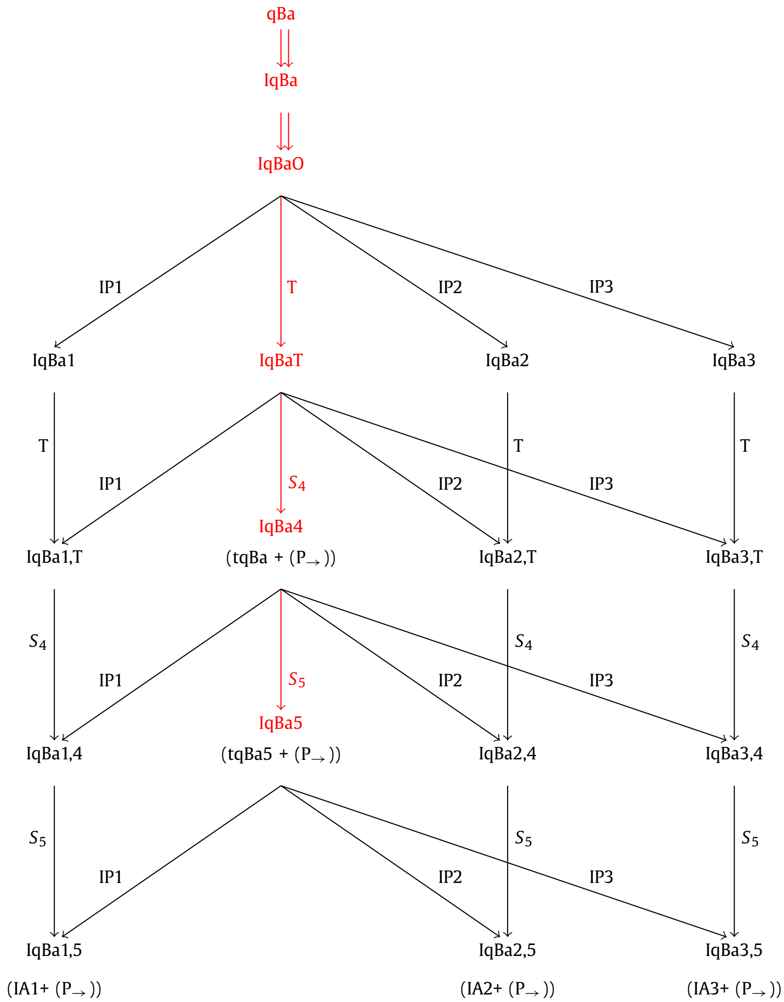

Fig. 1. Algebras based on IqBaO.
$P \rightrightarrows Q$ stands for the algebra Q has one more operation than the algebra P. $P \longrightarrow Q$ stands for both the algebras P and Q have the same operations but Q has one more axiom than P.

Definition 2.3. [10] An algebra $\langle S, \vee, \wedge, \rightarrow, \neg, O, 0, 1\rangle$, where O is a unary operator, will be called an implicative quasi-Boolean algebra with operator (IqBaO) if and only if

1. $\langle S, \vee, \wedge, \rightarrow, \neg, 0, 1\rangle$ is a IqBa.
2. $O1 = 1$.
3. $O(x \wedge y) = Ox \wedge Oy$, for all $x, y \in S$.

Definition 2.4. [10] Let $\langle S, \vee, \wedge, \rightarrow, \neg, O, 0, 1\rangle$ be a IqBaO. Then it will be an

1. implicative quasi-Boolean algebra with modal axiom T (IqBaT) if and only if $Ox \le x$ holds, for all $x \in S$ **(modal axiom T)**,
2. implicative quasi-Boolean algebra with modal axiom S_4 (IqBa4) if and only if it is a IqBaT and $Ox \le OOx$, for all $x \in S$ **(modal axiom S_4)**,
3. implicative quasi-Boolean algebra with modal axiom S_5 (IqBa5) if and only if it is a IqBa4 and $MOx \le Ox$, for all $x \in S$, where $M \equiv \neg O \neg$ **(modal axiom S_5)**.

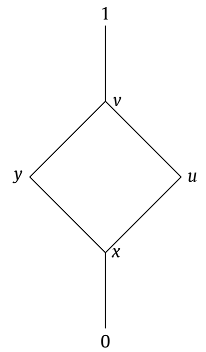

Fig. 2. Hasse diagram (IqBaO,IqBa2,IqBa4,IqBa5,IqBaT,IqBa3,T,IqBa3,4).

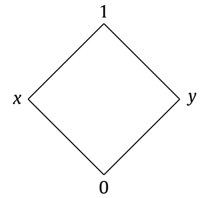

Fig. 3. Hasse diagram (IqBa1,IqBa3,IqBa1,4,IqBa1,5,IqBa3,5,tqBa5).

2.1. Algebras IqBa1, IqBaT, IqBa2, IqBa3 and their independence

It has been mentioned in the introduction that three intermediate properties IP1, IP2 and IP3 together play a crucial role. They are not independent as shown in [13,9] in the context of pre-rough algebra. We now add them to IqBaO separately and investigate the consequences. Before going to that we give an example of IqBaO where IP1, IP2 and IP3 do not hold.

Example 2.5. A lattice whose Hasse diagram is shown in Fig. 2 and $\to, \neg, O$ are defined below as

$\to$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	0	1	1	1	1	1
y	y	0	1	u	1	1
u	0	0	0	1	1	1
v	0	0	0	0	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	y	u	x	0
O	x	x	u	y	1	1

Clearly, it is a IqBaO. As $\neg Ox \vee Ox = v \neq 1$, IP1 does not hold. Again, $O(y \vee u) = 1 \neq Oy \vee Ou = v$, IP2 does also not hold. Here, $O1 \leq Ov$ and $M1 \leq Mv$ but $1 \nleq v$ and therefore IP3 does not hold.

Remark 2.6. T, S_4 and S_5 do also not hold in this example as $O0 \nleq 0$, $Oy = u \nleq OOy = y$, $MOy = y \nleq Oy = u$.

Definition 2.7. Let $\langle S, \vee, \wedge, \to, \neg, O, 0, 1\rangle$ be a IqBaO. Then it is said to be an

1. implicative quasi-Boolean algebra with IP1 (IqBa1) if and only if $\neg Ox \vee Ox = 1$ holds, for all $x \in S$,
2. implicative quasi-Boolean algebra with IP2 (IqBa2) if and only if $O(x \vee y) = Ox \vee Oy$ holds, for all $x, y \in S$,
3. implicative quasi-Boolean algebra with IP3 (IqBa3) if and only if $Mx \leq My$ and $Ox \leq Oy$ imply $x \leq y$, for all $x, y \in S$.

The following example shows that a IqBa1 is neither a IqBa2 nor a IqBa3 nor a IqBaT.

Example 2.8. A lattice whose Hasse diagram is shown in Fig. 3 and $\to, \neg, O$ are defined below as

$\to$	0	x	y	1
0	1	1	1	1
x	x	1	x	1
y	x	0	1	1
1	0	0	0	1

	0	x	y	1
$\neg$	1	y	x	0
O	0	y	0	1

Clearly, it is a IqBa1. As $O(x \vee y) = 1 \neq Ox \vee Oy = y$, IP2 does not hold. IP3 and T do not hold as $Oy \leq Ox$ and $My \leq Mx$ but $y \nleq x$ and $Ox \nleq x$.

Remark 2.9. S_4 and S_5 do not hold in this example as $Ox = y \nleq OOx = 0$, $MOx = x \nleq Ox = y$.

The following example shows that a IqBa2 is neither a IqBa1 nor a IqBa3 nor a IqBaT.

Example 2.10. A lattice whose Hasse diagram is shown in Fig. 2 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	0	1	1	1	1	1
y	0	0	1	0	1	1
u	0	0	0	1	1	1
v	0	0	0	0	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	u	y	x	0
O	x	x	x	y	y	1

Clearly, it is a IqBa2. As $\neg Ox \vee Ox = v \neq 1$, IP1 does not hold. IP3 and T do not hold here as $Oy \leq Ox$ and $My \leq Mx$ but $y \nleq x$ and $O0 = x \nleq 0$.

Remark 2.11. S_4 and S_5 do not hold in this example as $Ov = y \nleq OOv = x$, $MOv = u \nleq Ov = y$.

The following example shows that a IqBa3 is neither a IqBa1 nor a IqBa2 nor a IqBaT.

Example 2.12. A lattice is considered whose Hasse diagram is shown in Fig. 3 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	1
0	1	1	1	1
x	x	1	x	1
y	0	0	1	1
1	0	0	0	1

	0	x	y	1
$\neg$	1	x	y	0
O	0	y	0	1

It is a IqBa3. IP1, IP2 and T do not hold as $\neg Ox \vee Ox = y \neq 1$, $O(x \vee y) = 1 \neq Ox \vee Oy = y$, $Ox = y \nleq x$.

Remark 2.13. S_4 and S_5 do not hold in the above example as $Ox = y \nleq OOx = 0$, $MOx = 1 \nleq Ox = y$.

The following example is considered for an evidence of a IqBaT which is neither a IqBa1 nor a IqBa2 nor a IqBa3.

Example 2.14. A lattice whose Hasse diagram follows Fig. 2 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	0	1	1	1	1	1
y	0	x	1	0	1	1
u	0	0	0	1	1	1
v	0	v	0	0	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	y	u	x	0
O	0	x	x	x	y	1

Clearly, it is a IqBaT. IP1, IP2 and IP3 do not hold as $\neg Ox \vee Ox = v \neq 1$, $O(y \vee u) = y \neq Oy \vee Ou = x$, $Oy \leq Ou$ and $My \leq Mu$ but $y \nleq u$.

Remark 2.15. S_4 and S_5 do not hold in this example as $Ov = y \nleq OOv = x$, $MOx = y \nleq Ox = x$.

The above examples establish the independence of the algebras IqBaI, IqBa2, IqBa3 and IqBaT.

2.2. Algebras IqBa1,T, IqBa4, IqBa2,T, IqBa3,T and their independence

We now add the modal axiom T to each of the algebras IqBaI, IqBa2 and IqBa3.

Definition 2.16. Let $\langle S, \vee, \wedge, \rightarrow, \neg, O, 0, 1 \rangle$ be a IqBaO. Then it is said to be an

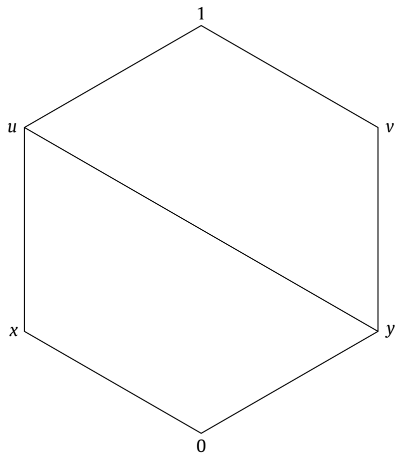

Fig. 4. Hasse diagram (IqBa1,T).

1. implicative quasi-Boolean algebra with IP1 and modal axiom T(IqBa1,T) if and only if it is a IqBa1 and $Ox \le x$, for all $x \in S$,
2. implicative quasi-Boolean algebra with IP2 and modal axiom T (IqBa2,T) if and only if it is a IqBa2 and $Ox \le x$, for all $x \in S$,
3. implicative quasi-Boolean algebra with IP3 and modal axiom T (IqBa3,T) if and only if it is a IqBa3 and $Ox \le x$, for all $x \in S$.

Example 2.8, Example 2.10 and Example 2.12 show that a IqBa1, a IqBa2, and a IqBa3 are not necessarily equal to a IqBa1,T, a IqBa2,T, and a IqBa3,T respectively.

The following example is considered to show that a IqBa1,T may not be a IqBa2,T, a IqBa3,T and a IqBa4.

Example 2.17. A lattice whose Hasse diagram is shown in Fig. 4 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	0	1	0	1	0	1
y	0	0	1	1	1	1
u	0	0	0	1	u	1
v	0	0	0	0	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	u	y	x	0
O	0	0	0	x	0	1

Obviously, it is a IqBa1,T but not a IqBa2,T or a IqBa3,T or a IqBa4 as $O(x \vee y) = x \neq Ox \vee Oy = 0$, $Ov \le Ou$ and $Mv \le Mu$ but $v \nleq u$, and $Ou = x \nleq OOu = 0$.

Remark 2.18. S_5 is also not available in Example 2.17 as $MOu = 1 \nleq Ou = x$.

The following example shows that a IqBa2,T may not be a IqBa1,T, a IqBa3,T and a IqBa4.

Example 2.19. A lattice whose Hasse diagram is shown in Fig. 5 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	x	1	1	1	1	1
y	0	0	1	1	1	1
u	0	0	0	1	1	1
v	0	v	0	0	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	u	y	x	0
O	0	0	x	x	y	1

Clearly, it is a IqBa2,T but neither a IqBa1,T nor a IqBa3,T nor a IqBa4 as $\neg Oy \vee Oy = v \neq 1$, $Ou \le Oy$ and $Mu \le My$ but $u \nleq y$ and $Oy = x \nleq OOy = 0$.

Remark 2.20. S_5 is also not available in Example 2.19 as $MOy = u \nleq Oy = x$.

Fig. 5. Hasse diagram (IqBa2,T,IqBa2,5).

The following example is considered to show that a IqBa3,T is neither a IqBa1,T nor a IqBa2,T nor a IqBa4.

Example 2.21. A lattice whose Hasse diagram is shown in Fig. 2 and $\to, \neg, O$ are defined below as

$\to$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	0	1	1	1	1	1
y	0	x	1	y	1	1
u	0	0	0	1	1	1
v	0	0	0	0	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	y	u	x	0
O	0	0	x	0	y	1

Obviously, it is a IqBa3,T but not a IqBa1,T, a IqBa2,T and a IqBa4 as $\neg O y \vee O y = v \neq 1$, $O(y \vee u) = y \neq O y \vee O u = x$, $O y = x \nleq O O y = 0$.

Remark 2.22. S_5 does not hold in this example as $M O y = y \nleq O y = x$.

The following example shows that a IqBa4 is neither a IqBa1,T nor a IqBa2,T nor a IqBa3,T.

Example 2.23. A lattice is considered whose Hasse diagram is shown in Fig. 2 and $\to, \neg, O$ are defined below as

$\to$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	0	1	1	1	1	1
y	0	x	1	0	1	1
u	0	0	0	1	1	1
v	0	0	0	v	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	y	u	x	0
O	0	0	0	0	v	1

Clearly, it is a IqBa4. As $\neg O v \vee O v = v \neq 1$, IP1 does not hold. IP2 and IP3 do not hold as $O(y \vee u) = v \neq O y \vee O u = 0$, $O y \leq O u$ and $M y \leq M u$ but $y \nleq u$.

Remark 2.24. S_5 does not hold in this example as $M O v = 1 \nleq O v = v$.

We notice from the above examples that the algebras IqBa1,T, IqBa2,T IqBa3,T and IqBa4 are independent.

2.3. Algebras IqBa1,4, IqBa5, IqBa2,4, IqBa3,4 and their independence

Modal axiom S_4 will now be added to each of the algebras IqBa1,T, IqBa2,T and IqBa3,T.

Definition 2.25. Let $\langle S, \vee, \wedge, \to, \neg, O, 0, 1\rangle$ be a IqBaO. Then it is said to be an

Fig. 6. Hasse diagram (IqBa2,4).

1. implicative quasi-Boolean algebra with IP1 and modal axiom S_4 (IqBa1,4) if and only if it is a IqBa1,T and $Ox \leq OOx$, for all $x \in S$,
2. implicative quasi-Boolean algebra with IP2 and modal axiom S_4 (IqBa2,4) if and only if it is a IqBa2,T and $Ox \leq OOx$, for all $x \in S$,
3. implicative quasi-Boolean algebra with IP3 and modal axiom S_4 (IqBa3,4) if and only if it is a IqBa3,T and $Ox \leq OOx$, for all $x \in S$.

Example 2.17, Example 2.19 and Example 2.21 show that IqBa1,T, IqBa2,T and IqBa3,T are not the same with IqBa1,4, IqBa2,4 and IqBa3,4 respectively.

The following example is of a IqBa1,4 but neither a IqBa2,4 nor a IqBa3,4 nor a IqBa5.

Example 2.26. A lattice whose Hasse diagram is shown in Fig. 3 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	1
0	1	1	1	1
x	0	1	x	1
y	0	0	1	1
1	0	0	0	1

	0	x	y	1
$\neg$	1	y	x	0
O	0	x	0	1

It is a IqBa1,4. As $MOx = 1 \nleq Ox = x$, it is not a IqBa5. IP2 and IP3 are not valid here as $O(x \vee y) = 1 \neq Ox \vee Oy = x$ and $Oy \leq Ox$ and $My \leq Mx$ but $y \nleq x$.

The following example is of a IqBa2,4 but neither a IqBa1,4 nor a IqBa3,4 nor a IqBa5.

Example 2.27. A lattice whose Hasse diagram is shown in Fig. 6 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	1
0	1	1	1	1
x	0	1	1	1
y	x	0	1	1
1	0	0	0	1

	0	x	y	1
$\neg$	1	y	x	0
O	0	x	x	1

It is a IqBa2,4. As $MOx = y \nleq Ox = x$, it is not a IqBa5. IP1 and IP3 are not valid in this example as $\neg Ox \vee Ox = y \neq 1$ and $Oy \leq Ox$ and $My \leq Mx$ but $y \nleq x$.

The following is an example of a IqBa3,4 which is neither a IqBa1,4 nor a IqBa2,4 nor a IqBa5.

Example 2.28. A lattice whose Hasse diagram is shown in Fig. 2 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	0	1	1	1	1	1
y	0	x	1	y	1	1
u	0	0	0	1	1	1
v	0	0	0	0	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	y	u	x	0
O	0	0	y	0	v	1

Clearly, it is a IqBa3,4. IP1, IP2 and S_5 do not hold here as $\neg Oy \vee Oy = y \neq 1$, $O(y \vee u) = v \neq Oy \vee Ou = y$ and $MOv = 1 \nleq Ov = v$.

The following is an example of a IqBa5 which is neither a IqBa1,4 nor a IqBa2,4 nor a IqBa3,4.

Example 2.29. A lattice whose Hasse diagram follows Fig. 2 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	0	1	1	1	1	1
y	0	x	1	0	1	1
u	0	0	0	1	1	1
v	0	0	0	v	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	y	u	x	0
O	0	x	x	x	v	1

Clearly, it is a IqBa5. IP1, IP2 and IP3 do not hold here as $\neg Ox \vee Ox = v \neq 1$, $O(y \vee u) = v \neq Oy \vee Ou = x$ and $Oy \leq Ou$ and $My \leq Mu$ but $y \nleq u$.

By the above examples, the independence of the algebras IqBa1,4, IqBa2,4, IqBa1,4 and IqBa5 is established.

2.4. Algebras IqBa1,5, IqBa2,5, IqBa3,5 and their independence

Similarly, adding the modal axiom S_5 to each of the algebras IqBaI,4, IqBa2,4 and IqBa3,4 we observe that new algebras so formed are not only independent to each other but not equivalent to previous one.

Definition 2.30. Let $\langle S, \vee, \wedge, \rightarrow, \neg, O, 0, 1\rangle$ be a IqBaO. Then it is said to be an

1. implicative quasi-Boolean algebra with IP1 and modal axiom S_5 (IqBa1,5) if and only if it is a IqBa1,4 and $MOx \leq Ox$, for all $x \in S$,
2. implicative quasi-Boolean algebra with IP2 and modal axiom S_5 (IqBa2,5) if and only if it is a IqBa2,4 and $MOx \leq Ox$, for all $x \in S$,
3. implicative quasi-Boolean algebra with IP3 and modal axiom S_5 (IqBa3,5) if and only if it is a IqBa3,4 and $MOx \leq Ox$, for all $x \in S$.

Example 2.26, Example 2.27 and Example 2.28 show that IqBa1,4, IqBa2,4 and IqBa3,4 are not the same with IqBa1,5, IqBa2,5 and IqBa3,5 respectively.

We now give an example of a IqBa1,5 which is neither a IqBa2,5 nor a IqBa3,5.

Example 2.31. Let us consider a lattice whose Hasse diagram is shown in Fig. 3 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	1
0	1	1	1	1
x	0	1	0	1
y	0	0	1	1
1	0	0	0	1

	0	x	y	1
$\neg$	1	y	x	0
O	0	0	0	1

Clearly, it is a IqBa1,5. IP2 and IP3 are not valid here as $O(x \vee y) = 1 \neq Ox \vee Oy = 0$ and $Ox \leq Oy$ and $Mx \leq My$ but $x \nleq y$.

The following example shows that a IqBa2,5 may not be a IqBa1,5 and a IqBa3,5.

Example 2.32. A lattice whose Hasse diagram is shown in Fig. 5 and $\rightarrow, \neg, O$ are defined below as

$\rightarrow$	0	x	y	u	v	1
0	1	1	1	1	1	1
x	x	1	1	1	1	1
y	0	0	1	1	1	1
u	0	0	0	1	1	1
v	0	0	v	0	1	1
1	0	0	0	0	0	1

	0	x	y	u	v	1
$\neg$	1	v	u	y	x	0
O	0	x	x	x	v	1

Clearly, it is a IqBa2,5 but neither a IqBa1,5 nor a IqBa3,5 as $\neg Ox \vee Ox = v \neq 1$, $Ou \leq Oy$ and $Mu \leq My$ but $u \nleq y$.

The following is an example of a IqBa3,5 which is neither a IqBa1,5 nor a IqBa2,5.

Example 2.33. A lattice whose Hasse diagram is shown in Fig. 3 and $\rightarrow, \neg, O$ are defined below as

Table 1
The algebras presented in Section 2.

Example	IP1	IP2	IP3	T	S_4	S_5	Type of algebra
Example 2.5	no	no	no	no	no	no	IqBaO
Example 2.8	yes	no	no	no	no	no	IqBa1
Example 2.10	no	yes	no	no	no	no	IqBa2
Example 2.12	no	no	yes	no	no	no	IqBa3
Example 2.14	no	no	no	yes	no	no	IqBaT
Example 2.17	yes	no	no	yes	no	no	IqBa1,T
Example 2.19	no	yes	no	yes	no	no	IqBa2,T
Example 2.21	no	no	yes	yes	no	no	IqBa3,T
Example 2.23	no	no	no	yes	yes	no	IqBa4
Example 2.26	yes	no	no	yes	yes	no	IqBa1,4
Example 2.27	no	yes	no	yes	yes	no	IqBa2,4
Example 2.28	no	no	yes	yes	yes	no	IqBa3,4
Example 2.29	no	no	no	yes	yes	yes	IqBa5
Example 2.31	yes	no	no	yes	yes	yes	IqBa1,5
Example 2.32	no	yes	no	yes	yes	yes	IqBa2,5
Example 2.33	no	no	yes	yes	yes	yes	IqBa3,5

$\to$	0	x	y	1
0	1	1	1	1
x	0	1	x	1
y	0	0	1	1
1	0	0	0	1

	0	x	y	1
$\neg$	1	x	y	0
O	0	x	0	1

Clearly, it is a IqBa3,5. IP1 and IP2 are not valid here as $\neg Ox \vee Ox = x \neq 1$ and $O(x \vee y) = 1 \neq Ox \vee Oy = x$.

Remark 2.34. The algebraic counterpart of the modal axiom K in the form $O(x \to y) \to (Ox \to Oy) = 1$ does not hold in the above examples except Example 2.31.

All examples stated above are either a IqBaO or IqBaO with some other additional properties. At a glance, from Table 1, one can see immediately which properties IP1, IP2, IP3, T, S_4 and S_5 hold in the above algebras. Here, 'yes' means the property holds and 'no' means the property does not hold. It is clear from this table that all algebras stated above are independent to each other.

Remark 2.35. It has been mentioned in [10] that the axiom $Ox \leq OOx$ is redundant in a IqBa5 and the modal axiom B ($MOx \leq x$) also holds in a IqBa5. Thus, the modal axiom S_4 is redundant in a IqBa1,5, IqBa2,5 and IqBa3,5 and the modal axiom B also holds in these algebras. Due to this reason, modal axiom S_4 is not considered as an axiom of the logic systems corresponding to the algebras IqBa1,5, IqBa2,5 and IqBa3,5 in Section 3.

It has been mentioned in [4] that in a tqBa no implication operator $\to$ satisfying the condition **($\mathbf{P}_\to$)** can in general be defined in terms of the other operations in it.

Example 2.36. (See [4]) A lattice whose Hasse diagram follows Fig. 3 and $\neg$, O are defined as

	0	x	y	1
$\neg$	1	x	y	0
O	0	x	y	1

It is a tqBa as well as tqba5. Now $x \to x$ must be an element involving $x, \neg, O, \vee, \wedge$ only. But it is observed that $\neg x = x$, $Ox = x$, $x \vee x = x$, $x \wedge x = x$ and hence $x \to x = x (\neq 1)$ whereas $x \leq x$.

With the above example it was mentioned in [10] that in IA2 and IA3 also no such $\to$ exists as it is also an instance of IA2 and IA3.

Remark 2.37. Whether such $\to$ exists or not in IA1 is unsolved.

Remark 2.38. As, in general, there is no such $\to$ defined in terms of other operations in tqBa, tqBa5, IA2 and IA3, in order to develop the Hilbert type axiomatic systems for the logics corresponding to these algebras one may impose $\to$ in these algebras in a general way satisfying the property **($\mathbf{P}_\to$)**. Then, a topological quasi-Boolean algebra with $\to$, i.e., a tqBa with $x \to y = 1$ if and only if $x \leq y$, for all x, y is identical with a IqBa4. Similarly, a topological quasi-Boolean algebra 5 with $\to$, Intermediate algebra of type 2 with $\rightarrow$ and Intermediate algebra of type 3 with $\rightarrow$ are identical with a IqBa5, IqBa2,5 and IqBa3,5 respectively.

It has been mentioned in the introduction that in [10], the authors introduced the notion of an implicative quasi-Boolean algebra (IqBa) following Rasiowa as, in general, no $\rightarrow$ was available in a qBa satisfying the property **($\mathbf{P}_{\rightarrow}$)**. Later, they added modal axioms T, S_4 and S_5 to obtain more structures. Following the same method we have added IP1, IP2 and IP3 separately to a IqBa in addition to these modal axioms T, S_4 and S_5 as, in general, no $\rightarrow$ was available satisfying the property **($\mathbf{P}_{\rightarrow}$)** in IA2, IA3 and in case IA1, it is unsolved till now. As a result, we have developed twelve structures in addition to the structures that are presented in [10]. Thus, this work may be considered as supplementary to [10].

3. The Hilbert-type axiomatic systems

We now present the Hilbert style systems corresponding to the algebras IqBa1, IqBa2, IqBa3, IqBa1,T, IqBa2,T, IqBa3,T, IqBa1,4, IqBa2,4, IqBa3,4, IqBa1,5, IqBa2,5 and IqBa3,5. For this, we follow the standard construction of a Lindenbaum-Tarski algebra for a given logic.

Lindenbaum-Tarski construction Let F be the set of all well formed formulae of a propositional logic that has a logical connective $\Rightarrow$. A relation $\approxeq$ on F is defined by $\gamma \approxeq \delta$ if and only if $\gamma \Rightarrow \delta$ and $\delta \Rightarrow \gamma$ are theorems where γ and δ are well formed formulae. Using the axioms and rules of the given logic it is proved that $\approxeq$ is an equivalence relation on F. The quotient set $F/\approxeq$ is then considered and a partial order relation $\leq$ is defined on it as $[\gamma] \leq [\delta]$ if and only if $\gamma \Rightarrow \delta$ is a theorem. It is shown that the equivalence relation $\approxeq$ on F is a congruence with respect to all logical connectives. The resulting algebra with the universe $F/\approxeq$ is called a Lindenbaum-Tarski algebra.

In [10], the Hilbert Systems L, L_O, L_T, L_4, L_5 corresponding to the algebras IqBa, IqBaO, IqBaT, IqBa4 and IqBa5 have been presented.

We call the logic systems for the algebras IqBa1, IqBa2, IqBa3, IqBa1,T, IqBa2,T, IqBa3,T, IqBa1,4, IqBa2,4, IqBa3,4, IqBa1,5, IqBa2,5 and IqBa3,5 as $L_1, L_2, L_3, L_{1,T}, L_{2,T}, L_{3,T}, L_{1,4}, L_{2,4}, L_{3,4}, L_{1,5}, L_{2,5}, L_{3,5}$ respectively.

3.1. Hilbert systems $L_1, L_{1,T}, L_{1,4}, L_{1,5}$

The alphabets of the language of all the systems $L_1, L_{1,T}, L_{1,4}, L_{1,5}$ consist of

- propositional variables r, s, t,...
- two unary logical connectives $\backsim$ and I.
- two binary logical connectives $\wedge$ and $\Rightarrow$.
- parentheses (,).

All well formed formulas (wffs) are formed as usual way and we denote them as $\alpha, \beta, \gamma, \delta$ etc.

Two definable logical connectives $\vee$ (binary) and C (unary) are defined as $\gamma \vee \delta \equiv \backsim (\backsim \gamma \wedge \backsim \delta), C\gamma \equiv \backsim I \backsim \delta$.

Axioms for L_1:

1. $\gamma \Rightarrow \backsim\backsim \gamma$
2. $\backsim\backsim \gamma \Rightarrow \gamma$
3. $\gamma \wedge \delta \Rightarrow \delta$
4. $\gamma \wedge \delta \Rightarrow \delta \wedge \gamma$
5. $\gamma \wedge (\delta \vee \beta) \Rightarrow (\gamma \wedge \delta) \vee (\gamma \wedge \beta)$
6. $(\gamma \wedge \delta) \vee (\gamma \wedge \beta) \Rightarrow \gamma \wedge (\delta \vee \beta)$
7. $I\gamma \wedge I\delta \Rightarrow I(\gamma \wedge \delta)$
8. $\backsim I\gamma \vee I\gamma$

Axioms for $L_{1,T}$: All axioms of L_1 along with one additional axiom (T): $I\gamma \Rightarrow \gamma$.
Axioms for $L_{1,4}$: All axioms of $L_{1,T}$ along with one additional axiom (S_4): $I\gamma \Rightarrow II\gamma$.
Axioms for $L_{1,5}$: All axioms of $L_{1,T}$ along with one additional axiom (S_5): $CI\gamma \Rightarrow I\gamma$.
The Rules of inference for $L_1, L_{1,T}, L_{1,4}, L_{1,5}$ are the same as prescribed below.

1. $\dfrac{\gamma, \gamma \Rightarrow \delta}{\delta}$ Modus ponens (MP)
2. $\dfrac{\gamma \Rightarrow \delta, \delta \Rightarrow \beta}{\gamma \Rightarrow \beta}$ Hypothetical syllogism (HS)
3. $\dfrac{\delta}{\gamma \Rightarrow \delta}$
4. $\dfrac{\gamma \Rightarrow \delta}{\backsim \delta \Rightarrow \backsim \gamma}$

5. $\dfrac{\gamma \Rightarrow \delta, \gamma \Rightarrow \beta}{\gamma \Rightarrow \delta \wedge \beta}$
6. $\dfrac{\gamma \Rightarrow \delta, \delta \Rightarrow \gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \alpha}{(\gamma \Rightarrow \alpha) \Rightarrow (\delta \Rightarrow \beta)}$
7. $\dfrac{\gamma \Rightarrow \delta}{I\gamma \Rightarrow I\delta}$
8. $\dfrac{\gamma}{I\gamma}$ Necessitation (N)

Definition 3.1. A model of $L_1/L_{1,T}/L_{1,4}/L_{1,5}$ is $\langle \mathbb{S}, V\rangle$ where $\mathbb{S} = \langle S, \vee, \wedge, \rightarrow, \neg, O, 0, 1\rangle$ is a IqBa1/IqBa1,T/IqBa1,4/IqBa1,5 and V is a valuation function which assigns a value $V(r) \in S$ for each atomic wff r of $L_1/L_{1,T}/L_{1,4}/L_{1,5}$.

Remark 3.2. Any valuation function V can be extended to arbitrary formulae as follows

$$V(\gamma \wedge \delta) = V(\gamma) \wedge V(\delta), V(\backsim \gamma) = \neg V(\gamma), V(\gamma \Rightarrow \delta) = V(\gamma) \rightarrow V(\delta), V(I\gamma) = OV(\gamma).$$

As $\vee$ and C are definable connectives, it can be shown that $V(\gamma \vee \delta) = V(\gamma) \vee V(\delta), V(C\gamma) = MV(\gamma)$ where $Mx = \neg O \neg x$.

Definition 3.3. A wff γ is said to be true in a model $\langle \mathbb{S}, V\rangle$ of $L_1/L_{1,T}/L_{1,4}$ $/L_{1,5}$ if and only if $V(\gamma) = 1$.

Definition 3.4. A wff γ is said to be valid in the class of all models of $L_1/L_{1,T}/L_{1,4}/L_{1,5}$ if and only if γ is true in every model $\langle \mathbb{S}, V\rangle$ of $L_1/L_{1,T}/L_{1,4}$ $/L_{1,5}$.

Remark 3.5. A wff $\gamma \Rightarrow \delta$ is valid if and only if $V(\gamma) \leq V(\delta)$, for all models $\langle \mathbb{S}, V\rangle$ of $L_1/L_{1,T}/L_{1,4}/L_{1,5}$.

Note 3.6. $\vdash \gamma$ stand for γ is a theorem in the logic system $L_1/L_{1,T}/L_{1,4}/L_{1,5}$ as usual sense.

Theorem 3.7. *(Soundness): If $\vdash \gamma$ in the logic system $L_1/L_{1,T}/L_{1,4}/L_{1,5}$ then γ is valid in the class of all models of $L_1/L_{1,T}/L_{1,4}/L_{1,5}$.*

Proof. (Outline) All axioms and rules of inference in the logic system $L_1/L_{1,T}$ $/L_{1,4}/L_{1,5}$ are valid. Using induction on the length of the proof of γ it can be established that γ is valid. □

Theorem 3.8. *(Completeness): If γ is valid in the class of all models of $L_1/L_{1,T}/L_{1,4}/L_{1,5}$ then $\vdash \gamma$ in the logic system $L_1/L_{1,T}/L_{1,4}/L_{1,5}$.*

Proof. (Outline) Here, we consider the logic system $L_{1,5}$ to prove the result. Proofs of other systems are similar. The Lindenbaum- Tarski algebra for the logic system $L_{1,5}$ with connectives $\wedge, \vee, \backsim, \Rightarrow, I, C$ is $\langle F/\approxeq, \vee, \wedge, \neg, \rightarrow, O, M\rangle$, where the operations $\vee, \wedge, \neg, \rightarrow, O, M$ are defined by the congruence relation $\approxeq$, i.e., $[\gamma] \vee [\delta] = [\gamma \vee \delta], [\gamma] \wedge [\delta] = [\gamma \wedge \delta], \neg[\gamma] = [\backsim \gamma], [\gamma] \rightarrow [\delta] = [\gamma \Rightarrow \delta], O[\gamma] = [I\gamma], M[\gamma] = [C\gamma]$. The partial order relation $\leq$, $[\gamma] \leq [\delta]$ if and only if $\vdash \gamma \Rightarrow \delta$, yields $\langle F/\approxeq, \leq, \vee, \wedge, \neg, \rightarrow, O, M, 0, 1\rangle$ as a IqBa1,5 where $0 = [\gamma], 1 = [\delta]$ such that $\vdash \backsim \gamma, \vdash \delta$. Now, we consider the valuation V such that $V(r) = [r]$, for all atomic wff $r \in F$. It can be extended over F as $V(\gamma) = [\gamma]$, for all $\gamma \in F$. Then, $\langle \mathbb{S}, V\rangle$ is a model of $L_{1,5}$ where $\mathbb{S} = \langle F/\approxeq, \leq, \vee, \wedge, \neg, \rightarrow, O, M, 0, 1\rangle$. As γ is valid in the class of all models of $L_{1,5}$, so it is true in the model $\langle \mathbb{S}, V\rangle$ and consequently $V(\gamma) = 1$ i.e., $\vdash \gamma$ in the logic system $L_{1,5}$. □

3.2. Hilbert systems $L_2, L_{2,T}, L_{2,4}, L_{2,5}$

The alphabets of the language of all the systems $L_2, L_{2,T}, L_{2,4}, L_{2,5}$ are the same as that of L_1.
Axioms for L_2: All axioms of L_1 except that 8 is replaced by $I(\gamma \vee \delta) \Rightarrow I\gamma \vee I\delta$.
Axioms for $L_{2,T}$: All axioms of L_2 along with one additional axiom (T): $I\gamma \Rightarrow \gamma$.
Axioms for $L_{2,4}$: All axioms of $L_{2,T}$ along with one additional axiom (S_4): $I\gamma \Rightarrow II\gamma$.
Axioms for $L_{2,5}$: All axioms of $L_{2,T}$ along with one additional axiom (S_5): $CI\gamma \Rightarrow I\gamma$.
The Rules of inference for $L_2, L_{2,T}, L_{2,4}, L_{2,5}$ are the same as that of L_1.

Theorem 3.9. *With respect to the corresponding algebras the above Hilbert Systems mentioned in Subsection 3.2 are sound and complete.*

Proof. All proofs are obtainable following Theorem 3.7 and Theorem 3.8. □

3.3. Hilbert systems $L_3, L_{3,T}, L_{3,4}, L_{3,5}$

The alphabets of the language of all the systems $L_3, L_{3,T}, L_{3,4}, L_{3,5}$ are the same as that of L_1.
Axioms for L_3: All axioms of L_1 except the axiom 8.
Axioms for $L_{3,T}$: All axioms of L_3 along with one additional axiom (T): $I\gamma \Rightarrow \gamma$.
Axioms for $L_{3,4}$: All axioms of $L_{3,T}$ along with one additional axiom (S_4): $I\gamma \Rightarrow II\gamma$.
Axioms for $L_{3,5}$: All axioms of $L_{3,T}$ along with one additional axiom (S_5): $CI\gamma \Rightarrow I\gamma$.
The Rules of inference for $L_3, L_{3,T}, L_{3,4}, L_{3,5}$ are all rules of L_1 along with one additional rule $\dfrac{I\gamma \Rightarrow I\delta, C\gamma \Rightarrow C\delta}{\gamma \Rightarrow \delta}$.

Theorem 3.10. *With respect to the class of corresponding algebras the above Hilbert Systems mentioned in Subsection 3.3 are sound and complete.*

Proof. All proofs are obtainable following Theorem 3.7 and Theorem 3.8. $\square$

4. Rough set models of some algebras with respect to the quasi-complementation

The notion of $quasi-complementation$ and $quasi-field\ of\ subsets\ of\ a\ set\ X$ has been discussed in [8]. In a non empty set X, the $quasi-complementation$ $\neg$ is defined by $\neg A = X - g(A) = g(A)^c$, for each $A \subseteq X$ where $g : X \to X$ is an involution, i.e., $g(g(a)) = a$, for all $a \in X$. It is obvious that every involution g is a bijective mapping on X. Moreover, for any involution $g : X \to X$ the following results hold.

- $g(g(A)) = A$, for all $A \subseteq X$.
- $g(A \cup B) = g(A) \cup g(B)$, for all $A, B \subseteq X$.
- $g(A \cap B) = g(A) \cap g(B)$, for all $A, B \subseteq X$.
- $\neg A = g(A)^c = g(A^c)$, for all $A \subseteq X$.
- $\neg\neg A = A$, for all $A \subseteq X$.
- $\neg(A \cap B) = \neg A \cup \neg B$, for all $A, B \subseteq X$.
- $\neg(A \cup B) = \neg A \cap \neg B$, for all $A, B \subseteq X$.

A collection $QF(X)$ of subsets of X, containing X and closed under set-theoretical union, intersection as well quasi-complementation $\neg$, $\langle QF(X), \cup, \cap, \neg, \emptyset, X\rangle$ is called a $quasi-field\ of\ subsets\ of\ X$. It has also been shown in [8] that quasi-fields of sets are typical examples of qBa, in the sense that every qBa is isomorphic to a quasi-field of sets.

For a non empty set X, thus, $\langle P(X), \cup, \cap, \neg, \emptyset, X\rangle$ becomes a qBa, where $P(X)$ is the power set of X and $\neg A = g(A)^c$ (g is an involution on X). It is to be noted that for an arbitrary involution g on X, $\langle P(X), \cup, \cap, \neg, \emptyset, X\rangle$ may not be a Boolean algebra as the quasi-complementation and complementation (set-theoretical) of a subset A of X are different in general, i.e., $\neg A (= g(A)^c) \neq A^c$ (see Example 4.11).

There are several definitions of Rough Sets of Pawlak [7,3]. Of these the most popular one is the lower-upper approximations pair definition. Given an approximation space (X, R), where X is a non empty set and R is an equivalence relation on it, a rough set is the pair $\langle \underline{A}, \overline{A}\rangle$ for $A \subseteq X$. $\underline{A} = \{x \in X : [x]_R \subseteq A\}$ and $\overline{A} = \{x \in X : [x]_R \cap A \neq \emptyset\}$; these sets are called the lower and upper approximations of the subset A of X relative to the approximation space. Rough set theory has been further generalised by an arbitrary relation in lieu of an equivalence relation [15,14,12] and by covering of a set [11,12]. For an arbitrary relation R on a non empty set X, it is known that the lower-upper approximations $\underline{A}_R = \{x \in X : R_x \subseteq A\}$ and $\overline{A}^R = \{x \in X : R_x \cap A \neq \emptyset\}$, where $R_x = \{y \in X : xRy\}$, are dual to each other with respect to the complementation, i.e., $\underline{(A^c)}_R = (\overline{A}^R)^c$ and $\overline{(A^c)}^R = (\underline{A}_R)^c$. But, in general, they are not dual with respect to the quasi-complementation as stated above, i.e., $\underline{(\neg A)}_R \neq \neg(\overline{A}^R)$ and $\overline{(\neg A)}^R \neq \neg(\underline{A}_R)$ (see Example 4.21).

As for any non empty set X, $\langle P(X), \cup, \cap, \neg, \emptyset, X\rangle$ is a model of qBa, so the quasi-complementation $\neg$ becomes an instance of $\neg$ available in qBa. Now, $\langle P(X), \cup, \cap, \neg, \emptyset, X\rangle$ is to be extended to a set theoretic rough set model containing dual operators O and M with respect to the quasi-complementation. For this, a pair of lower-upper approximations is defined so that they are dual with respect to the quasi-complementation. Due to this reason, we start with a generalised approximation space $\langle X, R\rangle$, where X is a non empty set and R is any relation on it and define g-lower and g-upper approximations so that they are dual with respect to the quasi-complementation $\neg$.

Let $\langle X, R\rangle$ be a generalised approximation space and $g : X \to X$ be an involution. A binary relation R^g on X is defined as follows: for any two elements x and y in X, $xR^g y$ if and only if $g(x)Rg(y)$. We now call $\langle X, R^g\rangle$ a g-generalised approximation space corresponding to the generalised approximation space $\langle X, R\rangle$ and the involution g on X or simply, a g-approximation space.

As g is an involution on X, R can be redefined with respect to R^g as follows: for any two elements x and y in X, xRy if and only if $g(x)R^g g(y)$. In general, there is no subset inclusion relation between R and R^g. However, the following results show how they are connected depending upon g.

Proposition 4.1. *In a g-approximation space $\langle X, R^g\rangle$, $R^g = R$ if and only if for all $x, y \in X$, xRy implies $g(x)Rg(y)$.*

Proof. Let $R^g = R$. Then, for any $x, y \in X$, xRy implies $xR^g y$ and hence $g(x)Rg(y)$ (by definition of R^g). Conversely, let for all $x, y \in X$, xRy imply $g(x)Rg(y)$. Let xRy. Then, $g(x)Rg(y)$ (by the hypothesis) and hence $xR^g y$ (by definition of R^g) and therefore $R \subseteq R^g$. Let $pR^g q$. Then, $g(p)Rg(q)$ (by definition of R^g) which implies $g(g(p))Rg(g(q))$ (by the hypothesis), i.e., pRq. Therefore, $R^g \subseteq R$. Thus, $R^g = R$. □

Remark 4.2. The following statements are equivalent in a g-approximation space $\langle X, R^g\rangle$.

1. $R^g = R$.
2. For all $x, y \in X$, $g(x)Rg(y)$ implies xRy.
3. For all $x, y \in X$, $xR^g y$ implies $g(x)R^g g(y)$.
4. For all $x, y \in X$, $g(x)R^g g(y)$ implies $xR^g y$.
5. $R \subseteq R^g$.
6. $R^g \subseteq R$.

Let $R_x = \{y \in X : xRy\}$ and $R^g_x = \{y \in X : xR^g y\}$. As before, there is no subset inclusion relation amongst $R_x, R_{g(x)}, R^g_x$ and $R^g_{g(x)}$ in general. But the following results show how they are related depending upon R and g.

Proposition 4.3. *In a g-approximation space $\langle X, R^g\rangle$, $R^g_x = R^g_{g(x)}$ ($R_x = R_{g(x)}$), for all $x \in X$ if and only if for all $x, y \in X$, $xR^g y$ (xRy) implies $g(x)R^g y$ ($g(x)Ry$).*

Proof. Let $R^g_x = R^g_{g(x)}$, for all $x \in X$. Then, for any $x, y \in X$, $xR^g y$ implies $y \in R^g_x = R^g_{g(x)}$ and hence $g(x)R^g y$. Conversely, let for all $x, y \in X$, $xR^g y$ imply $g(x)R^g y$. Let $y \in R^g_x$. Then, $xR^g y$ and hence $g(x)R^g y$ (by the hypothesis). This gives, $y \in R^g_{g(x)}$ and therefore $R^g_x \subseteq R^g_{g(x)}$. Let $p \in R^g_{g(x)}$. Then, $g(x)R^g p$ which implies $g(g(x))R^g p$ (by the hypothesis), i.e., $xR^g p$ and therefore $p \in R^g_x$. Thus, $R^g_{g(x)} \subseteq R^g_x$ and hence by earlier result $R^g_x \subseteq R^g_{g(x)}$ we get $R^g_x = R^g_{g(x)}$, for all $x \in X$.

The other case can be done similarly. □

Remark 4.4. The following statements are equivalent in a g-approximation space $\langle X, R^g\rangle$.

1. $R^g_x = R^g_{g(x)}$ ($R_x = R_{g(x)}$), for all $x \in X$.
2. $g(x)R^g y$ ($g(x)Ry$) implies $xR^g y$ (xRy), for all $x, y \in X$.
3. $R^g_x \subseteq R^g_{g(x)}$ ($R_x \subseteq R_{g(x)}$), for all $x \in X$.
4. $R^g_{g(x)} \subseteq R^g_x$ ($R_{g(x)} \subseteq R_x$), for all $x \in X$.
5. $R_x = R_{g(x)}$ ($R^g_x = R^g_{g(x)}$), for all $x \in X$.

Proposition 4.5. *In a g-approximation space $\langle X, R^g\rangle$, $R_x = g(R^g_{g(x)})$ and $R^g_x = g(R_{g(x)})$, for all $x \in X$.*

Proof. If $R_x = \emptyset$ then $R^g_{g(x)}$ is so. Also,

$$\begin{aligned} t \in R_x &\Leftrightarrow xRt \\ &\Leftrightarrow g(t) \in R^g_{g(x)} \\ &\Leftrightarrow g(x)R^g g(t) \\ &\Leftrightarrow g(g(t)) \in g(R^g_{g(x)}) \\ &\Leftrightarrow t \in g(R^g_{g(x)}) \end{aligned}$$

Similarly, the other holds. □

Proposition 4.6. *In a g-approximation space $\langle X, R^g\rangle$ the following results hold.*

1. *R^g is reflexive if and only if R is reflexive.*
2. *R^g is symmetric if and only if R is symmetric.*
3. *R^g is transitive if and only if R is transitive.*
4. *R^g is serial if and only if R is serial.*

Proof. (1). Let R be reflexive and $a \in X$. Then, there exists $b \in X$ such that $g(b) = a$ and bRb. Then, $g(b)R^g g(b)$ that is $aR^g a$. Let R^g be reflexive and $a \in X$. Then, $g(a)R^g g(a)$ and hence aRa.
Similarly, the others hold. $\square$

From the above proposition it follows that R^g is an equivalence relation on X if and only if R is so.

Proposition 4.7. *If R^g (R) is reflexive and transitive and $R^g_x = R^g_{g(x)}$ $(R_x = R_{g(x)})$, for all $x \in X$ then $R^g = R$.*

Proof. Let $xR^g y$. Then, $y \in R^g_x = R^g_{g(x)}$ and hence $g(x)R^g y$. As R^g is reflexive, $g(y) \in R^g_{g(y)} = R^g_y$ and therefore $yR^g g(y)$. By transitivity of R^g, $g(x)R^g g(y)$ and hence xRy and therefore $R^g \subseteq R$. Using Remark 4.2, $R^g = R$. $\square$

Remark 4.8.

1. The reflexivity and transitivity of $R^g(R)$ in the above proposition are necessary. If we drop any one of them then R^g and R may not be equal. The Example 4.9 is considered for an evidence of that.
2. The Example 4.10 shows that the converse of the above result is not true even for an equivalence relation R^g.

Example 4.9. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $g : X \to X$ be an involution defined by $g(1) = 4, g(2) = 6, g(3) = 3, g(4) = 1, g(5) = 5, g(6) = 2$. Let $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (2, 6), (6, 2), (3, 5), (5, 2)\}$ and $H = \{(3, 3), (5, 5), (3, 1)\}$. Then, $R^g = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (2, 6), (6, 2), (3, 5), (5, 6)\}$ is reflexive but not transitive and $H^g = \{(3, 3), (5, 5), (3, 4)\}$ is transitive but not reflexive. Here, $R^g_x = R^g_{g(x)}$ and $H^g_x = H^g_{g(x)}$, for all $x \in X$ but $R^g \neq R$ and $H^g \neq H$.

Example 4.10. X and g are the same as stated in Example 4.9. Let $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1)\}$. Then, $R^g = R$ and R^g is an equivalence relation on X but $R^g_2 = \{x \in X : 2R^g x\} = \{2\} \neq R^g_{g(2)} = \{6\}$.

The quasi-complementation and set theoretic complementation of a set A, i.e., $\neg A = g(A)^c$ and A^c are not the same even when R is an equivalence relation, $R = R^g$ and $R_x = R_{g(x)}$, for all $x \in X$. The following example establishes this.

Example 4.11. The same X and g as mentioned in Example 4.9 have been considered for this case also. Let $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (2, 6), (6, 2)\}$. Then, $R^g = R$, R^g is an equivalence relation on X and $R_x = R_{g(x)}$, for all $x \in X$. Let $A = \{1, 2, 4\}$. Then, $\neg A = g(A)^c = \{2, 3, 5\} \neq A^c = \{3, 5, 6\}$.

4.1. g-lower and g-upper approximations in a g-approximation space

Let $\langle X, R^g \rangle$ be a g-approximation space and A be any subset of X. $\underline{A}_g$, the g-lower approximation of A and $\overline{A}^g$, the g-upper approximation of A, in the g-approximation space $\langle X, R^g \rangle$, are defined by:

$$\underline{A}_g = \{x \in X : R^g_x \subseteq A\}$$

and

$$\overline{A}^g = \{x \in X : R^g_{g(x)} \cap g(A) \neq \emptyset\}.$$

Proposition 4.12. *The g-lower approximation $\underline{A}_g$ and g-upper approximation $\overline{A}^g$ are dual to each other with respect to the quasi-complementation $\neg$ defined through g.*

Proof.

$$\begin{aligned}
\neg\left(\underline{\neg A}_g\right) &= \neg\left(\underline{g(A)^c}_g\right) \\
&= \neg\{x \in X : R^g_x \subseteq g(A)^c\} \\
&= X - \{g(x) : R^g_x \subseteq g(A)^c\} \text{ [as } \neg A = X - g(A)] \\
&= X - \{x \in X : R^g_{g(x)} \subseteq g(A)^c\} \\
&= \{x \in X : R^g_{g(x)} \cap g(A) \neq \emptyset\} \\
&= \overline{A}^g.
\end{aligned}$$

As $\neg\neg A = A$, hence the result follows. $\square$

Proposition 4.13. *$\underline{A}_g$ and $\overline{A}^g$ are respectively $\underline{A}_{R^g}$ and $\overline{A}^R$.*

Proof. Clearly, $\underline{A}_g = \underline{A}_{R^g}$

$$\begin{aligned}\overline{A}^g &= \{x \in X : R^g_{g(x)} \cap g(A) \neq \emptyset\}\\ &= \{x \in X : g(R_x) \cap g(A) \neq \emptyset\} \text{ (by Proposition 4.5)}\\ &= \{x \in X : g(R_x \cap A) \neq \emptyset\}\\ &= \{x \in X : R_x \cap A \neq \emptyset\}\\ &= \overline{A}^R \quad \square\end{aligned}$$

Remark 4.14. It is noticeable from Example 4.21 that $\underline{A}_g \neq \underline{A}_R$ and $\overline{A}^g \neq \overline{A}^{R^g}$, even when R is an equivalence relation on X. Hence, for a subset A of X, $\langle \underline{A}_g, \overline{A}^g \rangle$ is different from $\langle \underline{A}_R, \overline{A}^R \rangle$ and $\langle \underline{A}_{R^g}, \overline{A}^{R^g} \rangle$. In fact, $\underline{A}_g$ is Pawlakian lower approximation of A in $\langle X, R^g \rangle$ and $\overline{A}^g$ is Pawlakian upper approximation of A in $\langle X, R \rangle$.

Note 4.15. In Proposition 4.13 we see that $\overline{A}^g = \overline{A}^R$. On the other hand, one may define $\underline{A}_g$ as Pawlakian lower approximation of A in $\langle X, R \rangle$, i.e., $\underline{A}_g = \underline{A}_R$. Then $\overline{A}^g$ (considering dual with respect to the quasi-complementation $\neg$) must be Pawlakian upper approximation of A in $\langle X, R^g \rangle$, i.e., $\overline{A}^g = \overline{A}^{R^g}$.

It has been mentioned earlier that $\underline{A}_g$ and $\overline{A}^g$ are dual approximations with respect to the quasi-complementation. But $\underline{A}_R$ and $\overline{A}^R$ are not so. In fact, they are dual approximations with respect to set theoretic complementation. We have established here a necessary and sufficient condition by which it can be checked whether for a given involution g on X, $\underline{A}_R$ and $\overline{A}^R$ are dual approximations with respect to quasi-complementation defined through g or not.

Theorem 4.16. *Let $\langle X, R \rangle$ be a generalised approximation space and g be an involution on X. Then for any $A \subseteq X$, $\underline{A}_R$ and $\overline{A}^R$ are dual approximations with respect to the quasi-complementation defined through g if and only if $R = R^g$.*

Proof. Let $R = R^g$. Then, $\underline{A}_g = \underline{A}_{R^g} = \underline{A}_R$ and $\overline{A}^g = \overline{A}^R$ and therefore $\underline{A}_R$ and $\overline{A}^R$ are dual approximations with respect to the quasi-complementation. Conversely, let $\underline{A}_R$ and $\overline{A}^R$ be dual approximations with respect to the quasi-complementation. Then, $\underline{A}_R = \neg(\overline{\neg A}^R)$. As, $\overline{A}^R = \overline{A}^g$, for all $A \subseteq X$. So, $\underline{A}_R = \neg(\overline{\neg A}^g) = \underline{A}_g = \underline{A}_{R^g}$ i.e., $\{x \in X : R_x \subseteq A\} = \{y \in X : R^g_y \subseteq A\}$, for all $A \subseteq X$. Let uRv. Then, $v \in R_u$. Let $A = R^g_u$. Then $\{x \in X : R_x \subseteq R^g_u\} = \{y \in X : R^g_y \subseteq R^g_u\}$ gives, $u \in \{x \in X : R_x \subseteq R^g_u\}$. Then, $R_u \subseteq R^g_u$. As $v \in R_u$, so $v \in R^g_u$ and hence $uR^g v$. Thus, $R \subseteq R^g$. Using Remark 4.2, $R = R^g$. $\square$

Remark 4.17. It is to be noted from Example 4.11 that the quasi- complementation and complementation of a set A i.e., $\neg A$ and A^c are not the same even when $R = R^g$. If they were the same, the above theorem would not have any significance at all.

However, when $R \neq R^g$, $\langle \underline{A}_g, \overline{A}^g \rangle \neq \langle \underline{A}_R, \overline{A}^R \rangle$ or $\langle \underline{A}_{R^g}, \overline{A}^{R^g} \rangle$, still the following results hold.

Proposition 4.18. *In a g-approximation space $\langle X, R^g \rangle$, the following results hold.*

1. $\underline{X}_g = X$ *and* $\overline{\emptyset}^g = \emptyset$.
2. *If* $A \subseteq B \subseteq X$ *then* $\underline{A}_g \subseteq \underline{B}_g$ *and* $\overline{A}^g \subseteq \overline{B}^g$.
3. $\underline{A \cap B}_g = \underline{A}_g \cap \underline{B}_g$ *and* $\overline{A \cup B}^g = \overline{A}^g \cup \overline{B}^g$*, for all* $A, B \subseteq X$.

Proposition 4.19. *If R^g is reflexive in a g-approximation space $\langle X, R^g \rangle$, the following results hold.*

1. $\overline{X}^g = X$ *and* $\underline{\emptyset}_g = \emptyset$.
2. $\underline{A}_g \subseteq A \subseteq \overline{A}^g$*, for all* $A \subseteq X$.

Proposition 4.20. *If R^g is transitive in a g-approximation space $\langle X, R^g \rangle$ then for any subset A of X, $\underline{A}_g \subseteq \underline{(\underline{A}_g)}_g$ and $\overline{(\overline{A}^g)}^g \subseteq \overline{A}^g$ hold.*

The following example is considered to show that $\overline{(\underline{A}_g)}^g \subseteq \underline{A}_g$ may not hold even for an equivalence relation R^g in a g-approximation space $\langle X, R^g\rangle$.

Example 4.21. X and g are the same as mentioned in Example 4.9. Let R be an equivalence relation on X which partitions the set X into the subsets $\{2,3\}, \{4\}, \{1,5\}, \{6\}$ of X. Then, the equivalence relation R^g partitions the set X into the subsets $\{3,6\}, \{1\}, \{4,5\}, \{2\}$ of X. Let $A = \{1,3,6\}$. Then, $\underline{A}_g = \{1,3,6\}$ and $\overline{(\underline{A}_g)}^g = \{1,2,3,5,6\}$ and therefore $\overline{(\underline{A}_g)}^g \nsubseteq \underline{A}_g$. Further, we see that $\underline{A}_g = \{1,3,6\} \neq \underline{A}_R = \{6\}$ and $\overline{A}^g = \{1,2,3,5,6\} \neq \overline{A}^{R^g} = \{1,3,6\}$. It is also noticeable that $\underline{A}_R$ and $\overline{A}^R$ are not dual approximations with respect to the quasi-complementation as $\underline{(\neg A)}_R = \{1,5,6\} \neq \neg(\overline{A}^R) = \{1\}$ and $\overline{(\neg A)}^R = \{1,5,6\} \neq \neg(\underline{A}_R) = \{1,3,4,5,6\}$.

Theorem 4.22. *Let R^g be an equivalence relation in a g-approximation space $\langle X, R^g\rangle$. Then for any subset A of X, $\overline{(\underline{A}_g)}^g \subseteq \underline{A}_g$ holds if and only if $R^g = R$.*

Proof. Let $R^g = R$. Then, $\langle \underline{A}_g, \overline{A}^g\rangle = \langle \underline{A}_R, \overline{A}^R\rangle$ and consequently for any subset A of X, $\overline{(\underline{A}_g)}^g \subseteq \underline{A}_g$ holds. Conversely, let $\overline{(\underline{A}_g)}^g \subseteq \underline{A}_g$ hold, for any subset A of X. Due to reflexivity of R^g, $\overline{(\underline{A}_g)}^g = \underline{A}_g$, for any subset A of X. Let xRy. Then, $[x]_R = [y]_R$ (as R is an equivalence relation), $[x]_R$ representing the equivalence class of x with respect to the relation R. Let $A = [x]_{R^g}$. Then, $\underline{A}_g = [x]_{R^g}$ and hence

$$\overline{[x]_{R^g}}^g = [x]_{R^g} \text{ (as } \overline{(\underline{A}_g)}^g = \underline{A}_g, \text{ for all } A \subseteq X),$$

$$\text{i.e., } \{z \in X : [z]_R \cap [x]_{R^g} \neq \emptyset\} = [x]_{R^g}. \tag{4.1}$$

Since $x \in [x]_R$ and $x \in [x]_{R^g}$, so $[x]_R \cap [x]_{R^g} \neq \emptyset$. As $[x]_R = [y]_R$, so $[y]_R \cap [x]_{R^g} \neq \emptyset$. It follows from (4.1) that $y \in \overline{[x]_{R^g}}^g = [x]_{R^g}$. This gives, $xR^g y$. Thus, $R \subseteq R^g$. Using Remark 4.2, $R^g = R$. □

By the above theorem it is clear that the counterpart of modal axiom S_5 is possible with respect to g-lower and g-upper approximations only when $R^g = R$. Indeed, in that case, g-lower and g-upper approximations are the same with Pawlakian lower and upper approximations in the approximation space $\langle X, R\rangle$. But one gain, in this case, is that $\underline{A}_g$ and $\overline{A}^g$ i.e., $\underline{A}_R$ and $\overline{A}^R$ are dual approximations with respect to the quasi-complementation. From Example 4.11, it is to be noted that complementation and quasi-complementation are not the same even when R is an equivalence relation.

4.2. Rough set models for IqBaO, IqBaT, IqBa4 and IqBa5

Rough Set model for IqBaO Let $\langle X, R^g\rangle$ be a g-approximation space. Now, $\langle P(X), \cup, \cap, \neg, \emptyset, X\rangle$ is a qBa, where $\neg A = g(A)^c$, for all $A \in P(X)$. We introduce $\rightarrow$ in $P(X)$ as follows
$A \rightarrow B = A^c \cup B$, for all $A, B \in P(X)$.
Then, it is obvious that $A \rightarrow B = X$ if and only if $A \subseteq B$ and consequently $\langle P(X), \cup, \cap, \rightarrow, \neg, \emptyset, X\rangle$ becomes a IqBa. We now define OA, for all $A \subseteq X$ as $OA = \underline{A}_g$. Then by Proposition 4.12 and Proposition 4.18, $\langle P(X), \cup, \cap, \rightarrow, \neg, O, \emptyset, X\rangle$ is a IqBaO.

Rough Set model for IqBaT For any reflexive relation R^g on X, by Proposition 4.12, Proposition 4.18 and Proposition 4.19, $\langle P(X), \cup, \cap, \rightarrow, \neg, O, \emptyset, X\rangle$ is a IqBaT.

Rough Set model for IqBa4 For any reflexive and transitive relation R^g on X, by Proposition 4.12, Proposition 4.18, Proposition 4.19 and Proposition 4.20, $\langle P(X), \cup, \cap, \rightarrow, \neg, O, \emptyset, X\rangle$ is a IqBa4.

Remark 4.23. If $\rightarrow$ is dropped from the above model of IqBa4 then $\langle P(X), \cup, \cap, \neg, O, \emptyset, X\rangle$ becomes a model of tqBa.

Rough Set model for IqBa5 For any equivalence relation R^g on X with $R^g = R$, by Proposition 4.12, Propositions 4.18, Proposition 4.19, Proposition 4.20 and Theorem 4.22, $\langle P(X), \cup, \cap, \rightarrow, \neg, O, M, \emptyset, X\rangle$ is a IqBa5 where $OA = \underline{A}_g = \underline{A}_R$ and $MA = \overline{A}^g = \overline{A}^R$.

Remark 4.24.

1. $\langle P(X), \cup, \cap, \neg, O, M, \emptyset, X\rangle$ becomes a model for a tqBa5 if $\rightarrow$ is dropped from the above model of IqBa5.
2. Defining $\underline{A}_g = \underline{A}_R$, $\overline{A}^g = \overline{A}^{R^g}$ and imposing the condition reflexivity/reflexivity and transitivity/equivalence on R it can be shown that $\langle P(X), \cup, \cap, \rightarrow, \neg, O_1, \emptyset, X\rangle$ is a model for IqBaO/IqBaT/IqBa4/IqBa5 where $O_1 A = \underline{A}_R$. Then, the two models with respect to the O and O_1 as stated above are different upto the algebra IqBa4 but the same for the algebra IqBa5.
3. If we define implication as $A \rightarrow_1 B = \neg A \cup g(B) = g(A \rightarrow B)$, for all $A, B \in P(X)$ then $\langle P(X), \cup, \cap, \rightarrow_1, \neg, O/O_1, \emptyset, X\rangle$ becomes a different model for IqBaO/IqBaT/IqBa4/IqBa5 with respect to the implication $\rightarrow_1$.

4.3. Rough set models for IqBa1, IqB1,T, IqB1,4 and IqB1,5

The following example is considered to show that $\neg(\underline{A}_g) \cup \underline{A}_g$ may not be the whole set X even when R^g is an equivalence relation on X and $R^g = R$.

Example 4.25. X, g and R are the same as stated in Example 4.10. Let $A = \{1, 2, 3\}$. Then, $\neg(\underline{A}_g) \cup \underline{A}_g = \{1, 2, 3, 4, 5\} \neq X$.

It is observed from the above Example 4.25 that $\underline{A}_g$ and $\overline{A}^g$ do not fit with IP1. So, we are now going to define a pair of new lower and upper approximations so that Rough set models for IqBa1, IqB1,T, IqB1,4 and IqB1,5 can be constructed.

Let $\langle X, R^g \rangle$ be a g-approximation space and A be any subset of X. $\underline{A}_{g,1}$, the $g,1$-lower approximation of A and $\overline{A}^{g,1}$, the $g,1$-upper approximation of A, in the g-approximation space $\langle X, R^g \rangle$, are defined by:

$$\underline{A}_{g,1} = \{x \in X : R^g_x \subseteq A\} \cap \{x \in X : R^g_{g(x)} \subseteq A\}$$

and

$$\overline{A}^{g,1} = \{x \in X : R^g_{g(x)} \cap g(A) \neq \emptyset\} \cup \{x \in X : R^g_x \cap g(A) \neq \emptyset\}.$$

Proposition 4.26. *The $g,1$-lower approximation $\underline{A}_{g,1}$ and $g,1$-upper approximation $\overline{A}^{g,1}$ are dual to each other with respect to the quasi-complementation $\neg$.*

Proof.

$$\begin{aligned}
\neg\left(\underline{\neg A}_{g,1}\right) &= \neg\left(\underline{g(A)^c}_{g,1}\right)\\
&= \neg\left(\{x \in X : R^g_x \subseteq g(A)^c\} \cap \{g(x) \in X : R^g_x \subseteq g(A)^c\}\right)\\
&= X - g\left(\{x \in X : R^g_x \subseteq g(A)^c\} \cap \{g(x) \in X : R^g_x \subseteq g(A)^c\}\right)\\
&= X - \left(\{g(x) \in X : R^g_x \subseteq g(A)^c\} \cap \{x \in X : R^g_x \subseteq g(A)^c\}\right)\\
&= \left(X - \{g(x) \in X : R^g_x \subseteq g(A)^c\}\right) \cup \left(X - \{x \in X : R^g_x \subseteq g(A)^c\}\right)\\
&= \left(X - \{x \in X : R^g_{g(x)} \subseteq g(A)^c\}\right) \cup \left(X - \{x \in X : R^g_x \subseteq g(A)^c\}\right)\\
&= \{x \in X : R^g_{g(x)} \cap g(A) \neq \emptyset\} \cup \{x \in X : R^g_x \cap g(A) \neq \emptyset\}\\
&= \overline{A}^{g,1}.
\end{aligned}$$

As $\neg\neg A = A$, result follows. □

Proposition 4.27. *$\underline{A}_{g,1}$ and $\overline{A}^{g,1}$ defined above are respectively $\underline{A}_{R^g} \cap g(\underline{A}_{R^g})$, i.e., $\underline{A}_g \cap g\left(\underline{A}_g\right)$ and $\overline{A}^R \cup g\left(\overline{A}^R\right)$, i.e., $\overline{A}^g \cup g\left(\overline{A}^g\right)$.*

Proof. $\underline{A}_{g,1} = \{x \in X : R^g_x \subseteq A\} \cap \{x \in X : R^g_{g(x)} \subseteq A\} = \{x \in X : R^g_x \subseteq A\} \cap \{g(x) \in X : R^g_x \subseteq A\} = \underline{A}_{R^g} \cap g\left(\underline{A}_{R^g}\right) = \underline{A}_g \cap g\left(\underline{A}_g\right)$.

$$\begin{aligned}
\overline{A}^{g,1} &= \{x \in X : R^g_{g(x)} \cap g(A) \neq \emptyset\} \cup \{x \in X : R^g_x \cap g(A) \neq \emptyset\}\\
&= \{x \in X : g(R_x \cap A) \neq \emptyset\} \cup \{x \in X : g(R_{g(x)} \cap A) \neq \emptyset\} \text{ (by Proposition 4.5)}\\
&= \{x \in X : R_x \cap A \neq \emptyset\} \cup \{x \in X : R_{g(x)} \cap A \neq \emptyset\}\\
&= \overline{A}^R \cup \{g(x) \in X : R_x \cap A \neq \emptyset\}\\
&= \overline{A}^R \cup g\left(\overline{A}^R\right)\\
&= \overline{A}^g \cup g\left(\overline{A}^g\right). \quad \square
\end{aligned}$$

Remark 4.28. For an arbitrary relation R^g, it follows from Proposition 4.27 and Proposition 4.13 that $\underline{A}_{g,1} \subseteq \underline{A}_g$ and $\overline{A}^R = \overline{A}^g \subseteq \overline{A}^{g,1}$, for all $A \subseteq X$.

Proposition 4.29. *If $R^g_x = R^g_{g(x)}$, for all $x \in X$ in a g-approximation space $\langle X, R^g \rangle$ then $\underline{A}_{g,1} = \underline{A}_g$ and $\overline{A}^{g,1} = \overline{A}^g$, for all $A \subseteq X$.*

Table 2
Three lower-upper approximations of a particular set.

	Case (i)	Case (ii)	Case (iii)	Case (iv)
A	$\{2,4\}$	$\{1,2,5\}$	$\{1,3,5\}$	$\{1,3,6\}$
$\langle \underline{A}_R, \overline{A}^R\rangle$	$\{4\}$, $\{2,3,4\}$	$\{1,5\}$, $\{1,2,3,5\}$	$\{1,5\}$, $\{1,2,3,5\}$	$\{6\}$, $\{1,3,5,6\}$
$\langle \underline{A}_g, \overline{A}^g\rangle$	$\{2\}$, $\{2,3,4\}$	$\{1,2\}$, $\{1,2,3,5\}$	$\{1\}$, $\{1,2,3,5\}$	$\{1,3,6\}$, $\{1,2,3,5,6\}$
$\langle \underline{A}_{g,1}, \overline{A}^{g,1}\rangle$	$\emptyset$, $\{1,2,3,4,6\}$	$\emptyset$, X	$\emptyset$, X	$\{3\}$, X
Remark	$\underline{A}_R$ and $\underline{A}_g$ have no common intersection	$\underline{A}_R$ and $\underline{A}_g$ have a non-void intersection	$\underline{A}_g$ is a proper subset of $\underline{A}_R$	$\underline{A}_R$ is a proper subset of $\underline{A}_g$

In the following example we have shown how the three pairs $\langle \underline{A}_R, \overline{A}^R\rangle$, $\langle \underline{A}_g, \overline{A}^g\rangle$ and $\langle \underline{A}_{g,1}, \overline{A}^{g,1}\rangle$ of a particular set look like when R is an equivalence relation, $R \neq R^g$ and $R_x \neq R_{g(x)}$, for at least one $x \in X$.

Example 4.30. X, g and R are the same as stated in Example 4.21. The possible cases are presented in Table 2.

It has been mentioned earlier that for any relation R, $\underline{A}_{g,1} \subseteq \underline{A}_g$ and $\overline{A}^R = \overline{A}^g \subseteq \overline{A}^{g,1}$ hold. As there is no fixed subset inclusion relation between $\underline{A}_R$ and $\underline{A}_g$ until $R = R^g$, the four cases that we have shown in Table 2 are the only possible cases when $\underline{A}_R \neq \underline{A}_g$. A Pictorial representation of these four cases are shown in more general way in Fig. 7.

Proposition 4.31. *In a g-approximation space $\langle X, R^g\rangle$, the following results hold.*

1. $\underline{X}_{g,1} = X$ *and* $\overline{\emptyset}^{g,1} = \emptyset$.
2. *If* $A \subseteq B \subseteq X$ *then* $\underline{A}_{g,1} \subseteq \underline{B}_{g,1}$ *and* $\overline{A}^{g,1} \subseteq \overline{B}^{g,1}$.
3. $\underline{A \cap B}_{g,1} = \underline{A}_{g,1} \cap \underline{B}_{g,1}$ *and* $\overline{A \cup B}^{g,1} = \overline{A}^{g,1} \cup \overline{B}^{g,1}$, *for all* $A, B \subseteq X$.
4. $\neg\left(\underline{A}_{g,1}\right) \cup \underline{A}_{g,1} = X$, *for all* $A \subseteq X$.

Proof. Proofs of first three are straightforward. For last one, if for any $x \in X$, $x \in \neg\left(\underline{A}_{g,1}\right)$, it is done. So, when $x \notin \neg\left(\underline{A}_{g,1}\right)$ then $x \in g(\underline{A}_{g,1}) = g\left(\underline{A}_{R^g} \cap g\left(\underline{A}_{R^g}\right)\right) = g(\underline{A}_{R^g}) \cap \underline{A}_{R^g} = \underline{A}_{g,1}$ □

Proposition 4.32. *If R^g is reflexive in a g-approximation space $\langle X, R^g\rangle$, the following results hold.*

1. $\overline{X}^{g,1} = X$ *and* $\underline{\emptyset}_{g,1} = \emptyset$.
2. $\underline{A}_{g,1} \subseteq A \subseteq \overline{A}^{g,1}$, *for all* $A \subseteq X$.

If R^g is transitive, even an equivalence relation, then $\underline{A}_{g,1} \subseteq \underline{(\underline{A}_{g,1})}_{g,1}$ may not hold. The example given below is one such.

Example 4.33. X and g are the same as stated in Example 4.9. Let $R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,2),(2,1)\}$. Then, $R^g = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(4,6),(6,4)\}$ is an equivalence relation on X. Let $A = \{1,4,6\}$. Then, $\underline{A}_{g,1} = \{1,4\}$ but $\underline{(\underline{A}_{g,1})}_{g,1} = \emptyset$.

Proposition 4.34. *If R^g is transitive and $R^g_x = R^g_{g(x)}$, for all $x \in X$ in a g-approximation space $\langle X, R^g\rangle$ then for any subset A of X, $\underline{A}_{g,1} \subseteq \underline{(\underline{A}_{g,1})}_{g,1}$ and $\overline{(\overline{A}^{g,1})}^{g,1} \subseteq \overline{A}^{g,1}$ hold.*

Proof. From Proposition 4.29, $\underline{A}_{g,1} = \underline{A}_g$ and $\overline{A}^{g,1} = \overline{A}^g$. As R^g is transitive, result follows from Proposition 4.20. □

Remark 4.35. The condition as stated in the above proposition is a sufficient condition but not necessary. The following example establishes that for any subset A of X, $\underline{A}_{g,1} \subseteq \underline{(\underline{A}_{g,1})}_{g,1}$ holds where R^g is transitive, even an equivalence relation, but $R^g_x \neq R^g_{g(x)}$, for all $x \in X$.

Example 4.36. Let $X = \{a, b\}$ and $g : X \to X$ be an involution defined by $g(a) = b$, $g(b) = a$. Let $R = \{(a,a),(b,b)\}$. Then $R^g = \{(b,b),(a,a)\}$ is an equivalence relation on X. Here, $\underline{A}_{g,1} = \underline{(\underline{A}_{g,1})}_{g,1}$, for all subset A of X but $R^g_a \neq R^g_{g(a)}$ and $R^g_b \neq R^g_{g(b)}$.

Table 3
Some important results on the three lower-upper approximations.

Nature of R	Result
R is arbitrary relation	$\underline{A}_g, \overline{A}^g$ and $\underline{A}_{g,1}, \overline{A}^{g,1}$ are always dual approximations with respect to the quasi-complementation.
R is arbitrary, $R \neq R^g$ and $R_x \neq R_{g(x)}$, for at least one $x \in X$	(1) $\underline{A}_{g,1} \subseteq \underline{A}_g$. (2) $\overline{A}^R = \overline{A}^g \subseteq \overline{A}^{g,1}$. (3) $\underline{A}_R$ and $\overline{A}^R$ are not dual approximations with respect to the quasi-complementation.
R is reflexive/equivalence, $R \neq R^g$ and $R_x \neq R_{g(x)}$, for at least one $x \in X$	(1) $\underline{A}_{g,1} \subseteq \underline{A}_g \subseteq A \subseteq \overline{A}^g = \overline{A}^R \subseteq \overline{A}^{g,1}$. (2) $\underline{A}_R$ and $\overline{A}^R$ are not dual approximations with respect to the quasi-complementation. (3)$\underline{A}_R \subseteq A$ but there is no fixed subset inclusion relation between $\underline{A}_R$ and $\underline{A}_g$. See Table 2 and Fig. 7.
R is reflexive and transitive, $R_x = R_{g(x)}$, for all $x \in X$ and $R \neq R^g$	The case is not possible by Proposition 4.7.
R is arbitrary but not reflexive and transitive, $R_x = R_{g(x)}$, for all $x \in X$ and $R \neq R^g$	(1) $\underline{A}_{g,1} = \underline{A}_g$. (2) $\overline{A}^R = \overline{A}^g = \overline{A}^{g,1}$. (3) $\underline{A}_R$ and $\overline{A}^R$ are not dual approximations with respect to the quasi-complementation.
R is arbitrary, $R = R^g$ and $R_x \neq R_{g(x)}$, for at least one $x \in X$	(1) $\underline{A}_{g,1} \subseteq \underline{A}_g = \underline{A}_R$ (2)$\overline{A}^R = \overline{A}^g \subseteq \overline{A}^{g,1}$. (3) $\underline{A}_R$ and $\overline{A}^R$ are dual approximations with respect to the quasi-complementation.
R is reflexive/equivalence, $R = R^g$ and $R_x \neq R_{g(x)}$, for at least one $x \in X$	(1) $\underline{A}_{g,1} \subseteq \underline{A}_g = \underline{A}_R \subseteq A \subseteq \overline{A}^R = \overline{A}^g \subseteq \overline{A}^{g,1}$. (2) $\underline{A}_R$ and $\overline{A}^R$ are dual approximations with respect to the quasi-complementation.
R is arbitrary, $R = R^g$ and $R_x = R_{g(x)}$, for all $x \in X$	(1) $\underline{A}_{g,1} = \underline{A}_g = \underline{A}_R$ (2)$\overline{A}^R = \overline{A}^g = \overline{A}^{g,1}$. (3) $\underline{A}_R$ and $\overline{A}^R$ are dual approximations with respect to the quasi-complementation.

Proposition 4.37. *If R^g is an equivalence relation and $R^g_x = R^g_{g(x)}$, for all $x \in X$ in a g-approximation space $\langle X, R^g \rangle$ then for any subset A of X, $\overline{(\underline{A}_{g,1})}^{g,1} \subseteq \underline{A}_{g,1}$.*

The condition as stated in the above proposition is only sufficient. The example given below shows that for any subset A of X, $\overline{(\underline{A}_{g,1})}^{g,1} \subseteq \underline{A}_{g,1}$ holds for an equivalence relation R^g where $R^g_x \neq R^g_{g(x)}$, for all $x \in X$.

Example 4.38. X, g and R are the same as stated in Example 4.36. Here, $\overline{(\underline{A}_{g,1})}^{g,1} = \underline{A}_{g,1}$, for all subset A of X but $R^g_a \neq R^g_{g(a)}$ and $R^g_b \neq R^g_{g(b)}$.

Rough Set model for IqBa1 Let $\langle X, R^g \rangle$ be a g-approximation space. By Proposition 4.26 and Proposition 4.31, $\langle P(X), \cup, \cap, \rightarrow, \neg, O, \emptyset, X \rangle$ is a IqBa1 where $\neg A = g(A)^c$, $A \rightarrow B = A^c \cup B$ and $OA = \underline{A}_{g,1}$.

Rough Set model for IqBa1,T For any reflexive relation R^g on X, by Proposition 4.26, Proposition 4.31 and Proposition 4.32, $\langle P(X), \cup, \cap, \rightarrow, \neg, O, \emptyset, X \rangle$ is a IqBa1,T.

Rough Set model for IqBa1,4 For any reflexive and transitive relation R^g with $R^g_x = R^g_{g(x)}$, for all $x \in X$, by Proposition 4.26, Proposition 4.31, Proposition 4.32 and Proposition 4.34, $\langle P(X), \cup, \cap, \rightarrow, \neg, O, \emptyset, X \rangle$ is a IqBa1,4 where $OA = \underline{A}_{g,1} = \underline{A}_g = \underline{A}_R$ by Proposition 4.7.

Rough Set model for IqBa1,5 For any equivalence relation R^g on X with $R^g_x = R^g_{g(x)}$, for all $x \in X$, by Proposition 4.26, Proposition 4.31, Proposition 4.32, Proposition 4.34 and Proposition 4.37, $\langle P(X), \cup, \cap, \rightarrow, \neg, O, \emptyset, X \rangle$ is a IqBa1,5 where $OA = \underline{A}_{g,1} = \underline{A}_R$ and $MA = \overline{A}^{g,1} = \overline{A}^R$.

Remark 4.39.

1. $\langle P(X), \cup, \cap, \neg, O, M, \emptyset, X \rangle$ becomes a model of IP1 if $\rightarrow$ is dropped from the above model of IqBa1,5.
2. If implication is defined as $A \rightarrow_1 B = g(A \rightarrow B)$, for all $A, B \in P(X)$ then $\langle P(X), \cup, \cap, \rightarrow_1, \neg, I, \emptyset, X \rangle$ becomes a different model of IqBaO/IqBaT/IqBa4/IqBa5 with respect to the implication $\rightarrow_1$.

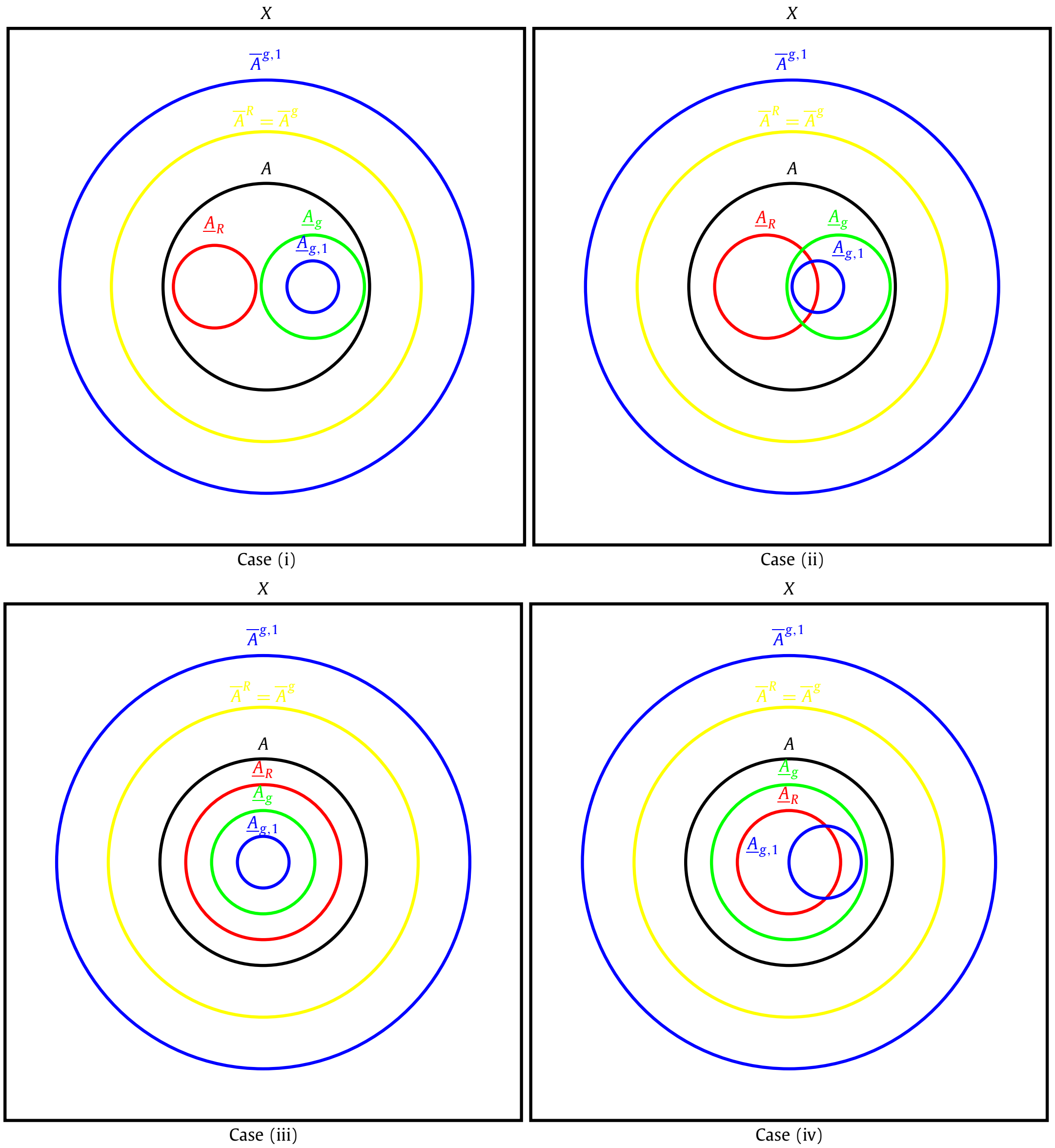

Fig. 7. Different possibilities of three lower-upper approximations when R is reflexive/equivalence, $R \neq R^g$ and $R_x \neq R_{g(x)}$, for at least one $x \in X$.

In order to view the important results of this section at a glance we refer to Table 3.

5. Concluding remarks

We may summarise the content and indicate some future directions of work of this paper as follows.

- First, we have developed three independent algebras by adding the three intermediate properties separately to an implicative quasi-Boolean algebra with operator. Further, a cluster of algebras, independent to each other, are obtained by adding modal axioms to them. A number of examples are considered to show their independence. Corresponding logic systems are developed and soundness-completeness theorems are established.
- In [10], Saha et al. introduced the notion of an implicative quasi-Boolean algebra following Rasiowa as no arrow can be defined in terms of other operations present in a qBa satisfying the property ($P_{\rightarrow}$). Afterwards, they added modal axioms and obtained more structures. In our case, we have followed the same basic principle and obtained twelve additional structures besides the structures that were developed in [10]. Thus, this work may be considered as supplementary to [10].

- Defining a new relation based on a generalised approximation space $\langle X, R\rangle$ and an involution g on X we have introduced two pairs of lower - upper approximations which are dual with respect to the quasi-complementation. Using them rough set models for algebras IqBaO, IqBaT, IqBa4, IqBa5, IqBa1, IqBa1,T, IqBa1,4, IqBa1,5, tqBa, tqBa5 and IA1 have been constructed. A necessary and sufficient condition is obtained when Pawlakian lower and upper approximations satisfy the notion of duality with respect to quasi-complementation.
- The natural question from the algebraic view point is about the representation theorem: Would the rough set models play the role of such representation for corresponding algebras? This point is for future investigation.
- Rough set model for the remaining algebras are still open for future work. A parallel study may be considered for covering case.
- We also pose the following question.
  Given a general approximation space $\langle X, R\rangle$, is it possible to construct an involution g depending on R such that $\underline{A}_R$ and $\overline{A}^R$ are dual approximations with respect to the quasi-complementation generated by g? Our guess is that it would be possible.

CRediT authorship contribution statement

Masiur Rahaman Sardar: Conceptualization, Methodology, Project administration, Writing – original draft, Writing – review & editing. **Mihir Kumar Chakraborty:** Conceptualization, Methodology, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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